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THE TEMPERATURE–VORTICITY ANALOGY IN BOUNDARY LAYERS†

W. J. McCROSKEY and S. H. LAM

Gas Dynamics Laboratory, Princeton University, Department of Aerospace and Mechanical Sciences, Princeton, New Jersey

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Abstract—Under certain generally-accepted assumptions, the analogy between temperature and vorticity for the viscous, two-dimensional flow of a fluid with constant properties can be extended to compressible boundary-layer flows. Within the scope of the assumptions, temperature is linearly related to shear stress, rather than to vorticity. The consequences of such a relationship are discussed for both laminar and turbulent flows. The temperature–shear-stress analogy is analyzed for the flow over a semi-infinite flat plate with a point heat source of strength Q located at the leading edge. Velocity and temperature profiles are presented for compressible laminar flow. They are non-similar, and depend upon the parameter

$$S = (Q/k_\infty T_\infty)R_x^{-\frac{1}{2}}.$$

Furthermore, they resemble the profiles of a compressible boundary layer on an insulated flat plate with $M_\infty^2 \propto S$ and with no heat source.

Experimental results were obtained which agree well with the theoretical calculations for laminar flows. However, the data do not follow the predictions of the temperature–vorticity analogy in transitional and turbulent-flow regimes, thereby demonstrating the inadequacy of the assumptions used in the theory. It was also found that the Reynolds number at which transition began could be adjusted by varying the strength of the heat source, with heat addition by means of the point source serving to delay transition to turbulence. It is suggested that there may be an important connection between the effects of Mach number and of point source heat addition upon the stability and transition characteristics of compressible boundary layers.

NOMENCLATURE

A, constant;
B, constant;
C_p, specific heat at constant pressure;
C_f, skin-friction coefficient;
f, dependent Blasius variable;
g, temperature function;
k, thermal conductivity;
M, Mach number;
n, exponent in viscosity law;
Pr, Prandtl number;
p, pressure;
Q, strength of the heat source;
R_x, Reynolds number, $\frac{\rho_\infty u_\infty x}{\mu_\infty}$;
S, the parameter $(Q/k_\infty T_\infty)R_x^{-\frac{1}{2}}$;
T, temperature;
t, time;
u, velocity parallel to the wall;
v, velocity normal to the wall;
x, longitudinal coordinate (in);
y, normal coordinate.

Greek symbols

γ, specific heat ratio;
δ, boundary-layer thickness;
δ^*, displacement thickness;
ζ, dummy variable of integration;
η, Blasius independent variable;
θ, momentum thickness;
μ, absolute viscosity;
ρ, density;
τ, shear stress;
ψ, stream function;
ω, vorticity.

Subscripts

∞, free stream properties;

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aw, adiabatic wall;
w, conditions at the wall;
T, transition;
0, zero-strength heat source.

1. INTRODUCTION

THERE IS a well-known analogy in the mechanics of viscous fluids between temperature and vorticity, for the two-dimensional flow of a fluid with constant properties. This paper is concerned with the validity of this analogy in laminar, transitional, and turbulent flows and the extension of the analogy to compressible fluids.

The temperature–vorticity analogy can be expressed in the form of an integral of the energy equation as follows. Let viscous dissipation of energy be neglected; then the vorticity diffusion and energy equations for the two-dimensional flow over an arbitrary body of a fluid with constant properties and Prandtl number equal to one are

$$\frac{d\omega}{dt} = \frac{\mu}{\rho}\nabla^2\omega \qquad (1)$$

$$\frac{dT}{dt} = \frac{\mu}{\rho}\nabla^2 T. \qquad (2)$$

Thus it is clear that an exact energy integral is

$$T = T_\infty + A\omega \qquad (3)$$

where A is a constant and T_∞ is the temperature of the free stream, which is assumed to be irrotational.

This relation between temperature and vorticity is well known and is cited in standard texts, such as Schlichting [1], to illustrate the nature of solutions to the Navier–Stokes equations in the limit of very small viscosity. Some additional comments have been made by Levy [2] and Meecham [3]. It is observed here that the analogy is based on the *time-dependent* Navier–Stokes equations. Therefore, equation (3) is formally valid for *all incompressible, two-dimensional laminar or turbulent flows*, provided the Prandtl number is one and the boundary conditions are satisfied.

The temperature–vorticity analogy can be extended to fluids with variable properties by applying the boundary-layer approximations to the Navier–Stokes equations. This has been noted by Lam [4] and Riley [5]. In the boundary-layer case, however, the particular solution for temperature is proportional to shear stress rather than to vorticity, and certain other restrictions are required. The temperature–shear-stress analogy in compressible boundary-layer flows is

$$T = T_\infty + B\tau \qquad (4)$$

where B is a constant. It should be noted that this analogy is distinct from the familiar Reynolds analogy, which is derived from the well-known Crocco integral and which relates shear stress to heat transfer.

The assumptions required for the temperature–shear-stress analogy are the following: constant pressure, Prandtl number equal to one, viscosity proportional to temperature, steady free-stream velocity, and two-dimensional flow. To test these assumptions and the validity of the analogy, an experimental and analytical study was made of the boundary layer on an insulated flat plate at zero angle of attack, with a point source of heat energy located at the leading edge, as shown in Fig. 1. From the results of the investigation, it has been concluded that the two-dimensional assumption is the most stringent one, and that the analogy probably breaks down in turbulent flow because of three-dimensional effects.

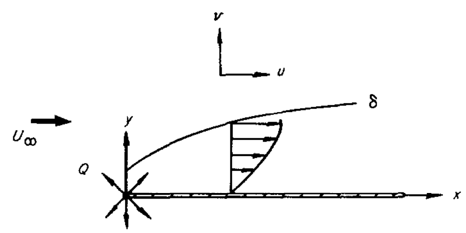

FIG. 1. Sketch of the physical problem. Q represents the strength of the heat source for the upper half plane.

During the course of the investigation, an unexpected phenomenon arose with regard to the stability of the boundary layer. The experiments showed that the transition of the laminar boundary layer to turbulent flow could be delayed by the addition of heat at the leading edge and that the transition Reynolds number could be adjusted to different values by varying the strength of the heat source. Since the details of the present solution at zero Mach number (which involves heating only by the point heat-source) are strikingly similar to well-known solutions for an insulated flat plate at $M > 0$, there are indications that there may be a close connection between the role of Mach number and the strength of the heat source in determining boundary layer stability and transition. Although this aspect of the investigation has not been resolved, it is discussed in the final section of the paper.

2. THEORY

There are two main objectives to be pursued before considering the results of the experiments. The first is to point out the distinction between the temperature–vorticity analogy and the temperature–shear-stress analogy. The second objective is to develop an exact solution for the laminar flow over an insulated flat plate with a point heat source at the leading edge.

The energy integrals $T = T_\infty + A\omega$ and $T = T_\infty + B\tau$ are both exact expressions and are closely related, although they apply under slightly different restrictions. The temperature–vorticity analogy applies to a two-dimensional flow of a constant-property fluid with Prandtl number equal to one. Relaxing the assumption of constant fluid properties requires additional assumptions, and the result is the temperature–shear-stress analogy. The temperature–shear-stress analogy requires the assumption of a constant-pressure, two-dimensional boundary-layer flow of a fluid with Prandtl number equal to one and viscosity proportional to temperature. Also, the free-stream velocity must be steady in both cases. The viscosity and Prandtl-number assumptions can be relaxed with little loss of generality; however, the two-dimensional assumption is essential.

For simplicity, a low-speed flow of a perfect gas with constant specific heats is assumed. Thus temperature variations which are due to viscous dissipation are not considered, and the conventional two-dimensional boundary-layer equations of continuity, momentum, and energy for constant Prandtl number become

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\left[\frac{1}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x}\right] \tag{6}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\rho Pr}\frac{\partial}{\partial y}\left(\mu\frac{\partial T}{\partial y}\right) + \frac{1}{\rho C_p}\frac{dp}{dt}. \tag{7}$$

Using the assumption $\mu \propto T$ and the boundary-layer approximation $\partial p/\partial y = 0$, one can transform the momentum equation by differentiation into

$$\frac{\partial \tau}{\partial t} + u\frac{\partial \tau}{\partial x} + v\frac{\partial \tau}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial \tau}{\partial y}\right) \tag{8}$$

where $\tau = \mu(\partial u/\partial y)$. Thus by inspection

$$T = T_\infty + B\tau \tag{4}$$

is a solution to the energy equation when $Pr = 1$ and $p =$ constant.

The appearance of the unsteady terms in equation (7) and (8) is very significant in that these equations account for *instantaneous* variations in temperature and vorticity, or more precisely, the quantity $\mu(\partial u/\partial y)$. Within the scope of the constant-pressure assumption and the related boundary-layer approximations, no statements are needed about whether the flow is laminar or turbulent. Therefore, temperature should be linearly related to shear stress in either type of flow, provided the boundary conditions are satisfied and the assumptions are valid. It should be emphasized that constant pressure is not required in the temperature–vorticity analogy.

The boundary conditions for the momentum equation are the classical ones,

$$u(x, \infty) = u_\infty = \text{constant}$$
$$u(x, 0) = 0.$$

Boundary conditions for the energy equation must be consistent with equation (4), and since $(\partial\tau/\partial y)_w = 0$, as shown below, it follows that $(\partial T/\partial y)_w = 0$. The use of a point source of heat at $x = 0$ is suggested by the well-known results for skin friction. At the outer edge of the boundary layer, $T = T_\infty$ = constant. As far as the unsteady boundary conditions are concerned, the flow is assumed to be laminar, and hence steady, at $x = 0$ and at $y \to \infty$. Fluctuations also vanish at the wall.

Attention is now directed toward the solution to the present physical problem, which is an insulated flat plate with a point heat source at the leading edge. For laminar flow, the conventional compressible stream function is introduced, defined by

$$u = \frac{\rho_\infty}{\rho}\frac{\partial\psi}{\partial y} \tag{9}$$

$$v = \frac{\rho_\infty}{\rho}\frac{\partial\psi}{\partial x} \tag{10}$$

and the following forms of solutions are assumed:

$$\psi = \left(\frac{\mu_\infty u_\infty x}{\rho_\infty}\right)^{\frac{1}{2}} f(\eta) \tag{11}$$

$$T = T_\infty + \frac{Q}{k_\infty Pr}\left(\frac{\rho_\infty u_\infty x}{\mu_\infty}\right)^{-\frac{1}{2}} g(\eta). \tag{12}$$

Here Q represents the strength of the heat source for the upper half of the plate, and has the dimensions of energy per unit time per unit width of the plate. The variable

$$\eta = \tfrac{1}{2}\left(\frac{\rho_\infty u_\infty}{\mu_\infty x}\right)^{\frac{1}{2}} \int_0^y \left(\frac{\rho}{\rho_\infty}\right) \mathrm{d}y \tag{13}$$

is a similarity variable in the transformed plane, where the velocity is given by

$$u = \tfrac{1}{2}u_\infty f'(\eta).\dagger \tag{14}$$

With the assumption $\mu \propto T$, the momentum and energy equations become, respectively,

$$\left.\begin{aligned} &f''' + ff'' = 0\\ &f(0) = 0\\ &f'(0) = 0\\ &f'(\infty) = 2 \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} &g'' + Pr(f'g + fg') = 0\\ &g'(0) = 0\\ &g(\infty) = 0. \end{aligned}\right\} \tag{16}$$

Equation (15) is the familiar Blasius equation. Equation (16) has the solution

$$g(\eta) = g(0)\left[\frac{f''(\eta)}{f''(0)}\right]^{Pr}. \tag{17}$$

Consideration of the enthalpy flux through a control volume enclosing the heat source and the leading edge of the plate, as suggested by Levy [2], gives

$$g(0) = \frac{f''^{Pr}(0)}{\int_0^\infty f'(\zeta)f''^{Pr}(\zeta)\,\mathrm{d}\zeta}. \tag{18}$$

When $Pr = 1$, $g(\eta) = \frac{1}{2}f''(\eta)$, as predicted by the temperature–shear-stress analogy.

The physical coordinate y can be determined from equations (12), (13), (17) and (18). Then in terms of the Blasius variable η and the parameter $S = (Q/k_\infty T_\infty)R_x^{-\frac{1}{2}}$, the results are

$$\frac{y}{\delta} = \frac{\eta + (S/Pr)\int_0^\eta g(\zeta)\,\mathrm{d}\zeta}{\eta_\delta + (S/Pr)\int_0^\infty g(\zeta)\,\mathrm{d}\zeta} \tag{19}$$

$$\frac{u}{u_\infty} = \tfrac{1}{2}f'(\eta) \tag{20}$$

$$\frac{T}{T_\infty} = 1 + \frac{S}{Pr}\frac{f''^{Pr}(\eta)}{\int_0^\infty f'(\zeta)f''^{Pr}(\zeta)\,\mathrm{d}\zeta}. \tag{21}$$

† Primes denote differentiation with respect to η.

Tabulated values of the Blasius function and its derivatives were used to calculate u/u_∞, T/T_∞, and y/δ, where the boundary-layer thickness δ is defined as the value of y for which $u/u_\infty = 0{\cdot}999$. The velocity and temperature profiles are shown in Figs. 2 and 3, respectively, for various values of S.

The velocity profiles for small values of S, representing a weak source or a point far downstream of the source, approach the classical Blasius profile. The deviations from the Blasius profile that are produced by the heat source for $S > 0$ are similar to the deviations produced by viscous dissipation, or aerodynamic heating, on an insulated flat plate with $M_\infty^2 \propto S$ and no heat source. They are very much alike, but not identical, because the curves of temperature distribution as a function of η are similar but not identical for the two cases, as shown in Fig. 4. In addition, the curves of velocity distribution as a function of η are identical. Since $y \propto \int T/T_\infty \, d\eta$ in each case, the velocity and temperature profiles at each value of S approach, though do not duplicate, velocity and temperature profiles at an appropriate Mach number which would give the same adiabatic wall temperature. For example, if $\gamma = 1{\cdot}4$ and $Pr = 1$, then the same insulated wall temperature would result if $M^2 = 3{\cdot}32S$.

The similarity of the two problems is evident from the tabulated expressions for velocity and temperature profiles, wall temperature, skin friction, displacement thickness, and momentum thickness that are shown in Table 1 for the case of $Pr = 1$. The velocity profiles of the present analysis for $Pr = 0{\cdot}75$ are compared in Fig. 5 with the results of calculations by van Driest [6] for aerodynamically heated boundary layers. Both solutions give the Blasius profile for $M = S = 0$. The curves at $M_\infty = 12$ ($T_{aw}/T_\infty = 25{\cdot}9$) and $S = 33$ ($T_{aw}/T_\infty = 25{\cdot}6$) are almost identical.

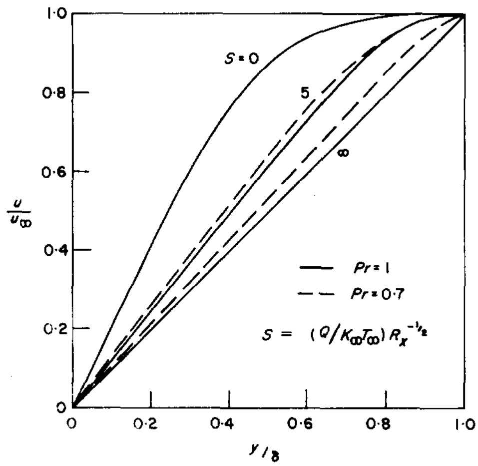

FIG. 2. Theoretical velocity profiles for various values of the parameter S.

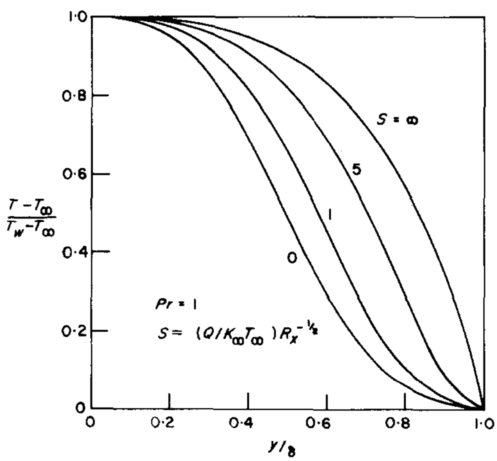

FIG. 3. Theoretical temperature profiles for various values of the parameter S.

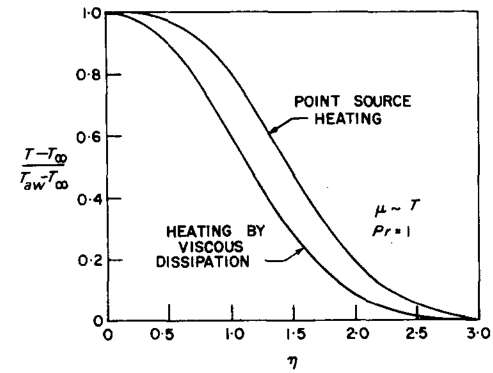

FIG. 4. Comparison of temperature profiles as a function of η on an insulated flat plate.

Table 1. Comparison of source heating and aerodynamic heating. Prandtl number equal to one, viscosity proportional to temperature

Point source heating	Aerodynamic heating
$\frac{u}{u_\infty} = \frac{1}{2}f'(\eta)$	$\frac{u}{u_\infty} = \frac{1}{2}f'(\eta)$
$\frac{T}{T_\infty} = 1 + \frac{1}{2}Sf''(\eta)$	$\frac{T}{T_\infty} = 1 + \frac{\gamma - 1}{2}M_\infty^2[1 - \frac{1}{4}f'^2(\eta)]$
$\frac{T_{aw} - T_\infty}{T_\infty} = 0{\cdot}664S$	$\frac{T_{aw} - T_\infty}{T_\infty} = \frac{\gamma - 1}{2}M_\infty^2$
$C_f = \frac{0{\cdot}664}{\sqrt{R_x}}$	$C_f = \frac{0{\cdot}664}{\sqrt{R_x}}$
$\delta^* = \frac{x}{\sqrt{R_x}}[2S + 1{\cdot}72]$	$\delta^* = \frac{x}{\sqrt{R_x}}\left[2{\cdot}38\frac{\gamma - 1}{2}M_\infty^2 + 1{\cdot}72\right]$
$\theta = 0{\cdot}664\frac{x}{\sqrt{R_x}}$	$\theta = 0{\cdot}664\frac{x}{\sqrt{R_x}}$
$S = \frac{Q}{k_\infty T_\infty}R_x^{-\frac{1}{2}}$	

An aspect of the theory that is related to this discussion of velocity profiles is the prediction of an increment in boundary-layer thickness which is independent of the distance from the source. This increment is in addition to the incompressible boundary-layer thickness δ_i, which is proportional to x. For $Pr = 1$, the boundary-layer thickness is

$$\delta = \delta_i + \frac{2Q}{\rho_\infty u_\infty C_p T_\infty}. \tag{22}$$

It is the latter term in equation (22) which alters the theoretical velocity profiles from the Blasius profile for $S = 0$ ($x \to \infty$ or $Q = 0$) to a linear profile at $x = 0$. Of course, the boundary-layer approximations are not strictly valid at $x = 0$, but the linear profile represents the limiting results.

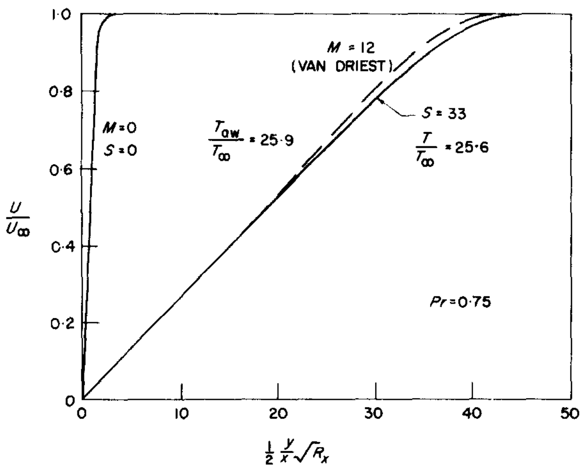

FIG. 5. Comparison of velocity profiles on an insulated flat plate.

3. DESCRIPTION OF THE EXPERIMENT

The experimental program was directed toward verifying the dependence of wall temperature upon the parameter $S = (Q/k_\infty T_\infty)R_x^{-\frac{1}{2}}$ in

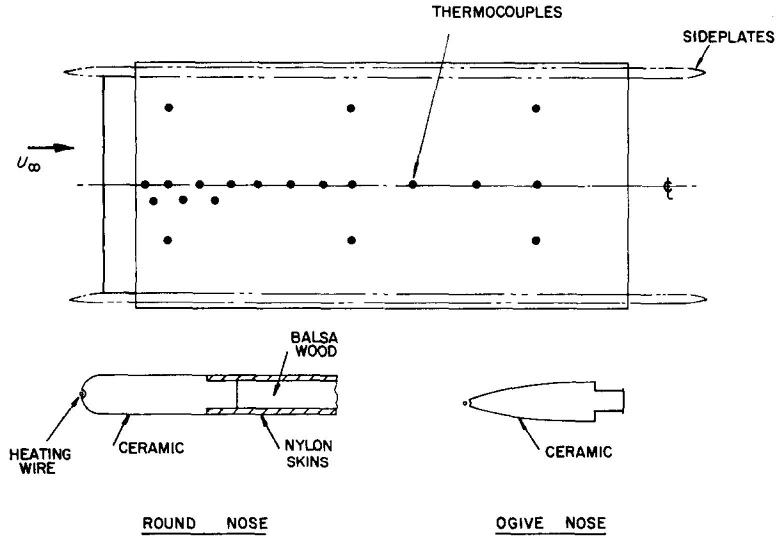

FIG. 6. Model geometry.

laminar flow, investigating the qualitative predictions of the analogy in transitional and turbulent-flow regimes, and investigating variations in transition Reynolds number. Reynolds number was varied from about 3500 to 350000 and the source-strength parameter $Q/k_\infty T_\infty$ from zero to about 40.

The theoretical problem was represented by an insulated flat-plate model with an electrically heated resistance wire spanning the leading edge. Side plates were used to help maintain two-dimensional flow. The model, heater, and side plates are shown in Fig. 6. The model consisted of thin nylon skins bonded to a sheet of balsa plywood, with a nose piece made from a high-temperature ceramo-plastic. A simplified analysis of the heat conduction within the model indicated negligible errors in the wall temperature for the nylon–balsa construction, despite the large longitudinal temperature gradient. Two nose geometries, hemi-cylindrical and ogival, were used to study pressure gradient and transition effects. Transition Reynolds number was found to be an order of magnitude greater for the ogive nose; otherwise, no differences were noticed in the results.

Wall temperatures were measured by fine thermocouple wires imbedded in small cavities in the nylon skins at the locations shown in Fig. 6. The cavities were filled with metallic circuit paint to form a smooth and flush surface. The reference junction was placed in the free stream so that $(T_w - T_\infty)$ was measured directly. The two-dimensional point heat source was in effect replaced by a three-dimensional line source whose strength was assumed constant along its length. However, end losses, as well as radiation losses, were considered in the calculation of the strength of the heat source. These losses ranged from about 5 per cent to 15 per cent of the source strength. Two sizes of round wires and a thin flat ribbon were used as heaters, with no detectable difference in the results.

The model was tested in the Princeton University 3 ft × 4 ft subsonic smoke tunnel. Standard hot-wire anemometry was used for the detection of turbulent fluctuations in the boundary layer. Electrical measurements of heater resistance and current were used to determine the total heat input and the temperature of the heater wire. Wall temperature and total heater power measurements were normally recorded under steady-free-stream conditions, and the hot-wire anemometry was used to determine the laminar or turbulent state of the boundary layer.

Transition to turbulence was studied for both natural and forced transition. Transition was forced by turbulence grids in front of the model and by single element and distributed roughness on the surface of the model. The single-element trip consisted of a piece of serrated masking tape spanning the width of the model and fixed the point of transition virtually independent of velocity. The distributed roughness consisted of grit particles varying in size from about 0·003-in diameter near the nose of the model to about 0·010-in diameter near the rear of the model. This distribution was only partially successful in fixing transition Reynolds number for $Q = 0$, as shown in Fig. 7.

In some instances, wall-temperature data were recorded with and without the trip elements or turbulence grids so as to compare laminar and turbulent data for identical values of the parameter $S = (Q/k_\infty T_\infty)R_x^{-\frac{1}{2}}$. In such cases, removal of the trip tape or screen was accomplished during the run from outside the test section, without changing tunnel speed or heater power.

Tests were also conducted in which the transition Reynolds number was varied by changing the source strength parameter $Q/k_\infty T_\infty$. The transition Reynolds number, taken to be the point at which the first bursts of turbulent fluctuations were observed, was first determined for $Q = 0$. Next Q was increased until the boundary layer became completely laminar. Then holding Q fixed, R_x was increased, either by increasing u_∞ while keeping the x-position of the hot wire fixed or by holding u_∞ constant and increasing x, until the boundary layer attained a transitional state. Then holding R_x constant, Q was increased until the boundary layer again became laminar. Repetition of this alternating process yielded the narrow transition band in the R_x vs. S plane that is shown in Fig. 8. From

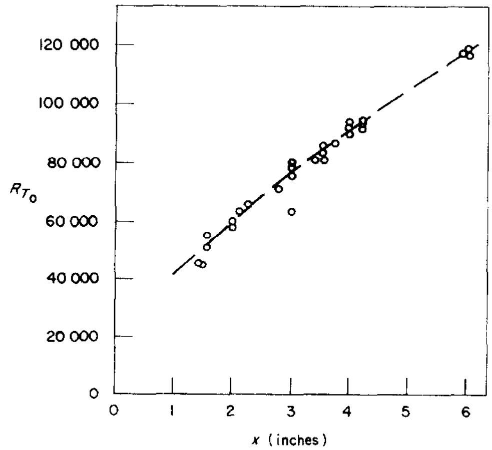

FIG. 7. Variation of transition Reynolds number along the plate with zero heat addition and uniformly distributed roughness.

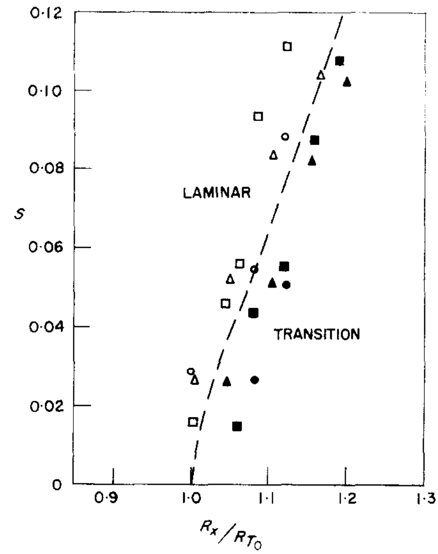

FIG. 8. Transition data for the rough plate.

these and other similar data, transition Reynolds numbers could be estimated at appropriate values of $Q/k_\infty T_\infty$.

4. RESULTS AND DISCUSSION

Temperature, vorticity, and shear stress

As stated earlier, the well-known energy integral $T = T_\infty + A\omega$ is an exact expression for both laminar and turbulent flows of a constant-property fluid, if the Prandtl number is one and the motion is two-dimensional. The present experiments were performed in air in order to approximate the Prandtl-number restriction, and as a consequence, the fluid properties were not strictly constant. Therefore, the quantitative comparisons are to be made with the theoretical predictions of the temperature–shear-stress analogy. The qualitative behavior would be expected to be generally the same for either case.

Laminar values of experimental wall temperature data are plotted in Figs. 9 and 10. The good agreement with equation (21) verifies the temperature–shear-stress analogy for laminar flow.

However, the analogy fails in the transition to turbulent flow. Skin friction and vorticity are generally acknowledged to increase through the transition to turbulence. Since $(T - T_\infty)$ is theoretically proportional to ω (or to τ, if ρ and μ are variable) even in the unsteady case, then it follows that the wall temperature should also increase as the boundary layer goes turbulent. However, the experimental results in Figs. 10–13 show the opposite effect of turbulence on wall temperature, in both natural and forced transition.

The test which produced the data in Fig. 13

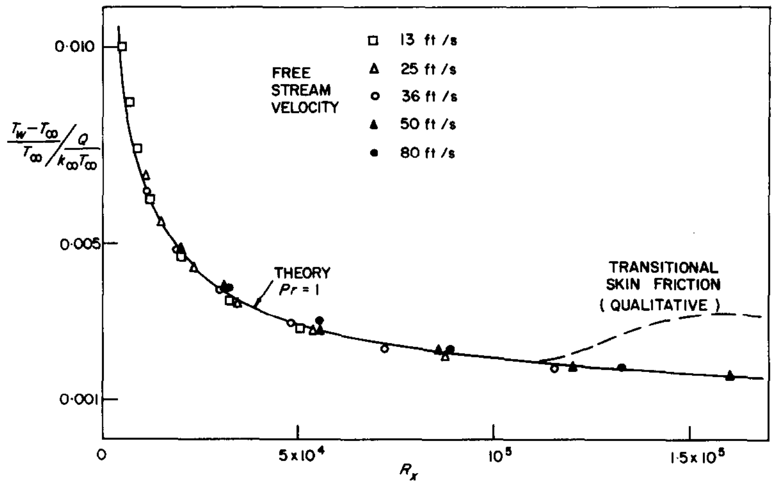

FIG. 9. Wall temperature vs. Reynolds number for laminar boundary layers.

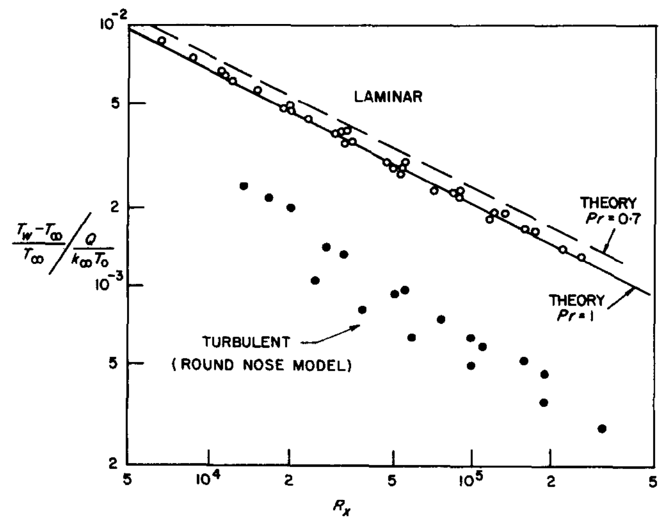

FIG. 10. Laminar and turbulent wall-temperature data.

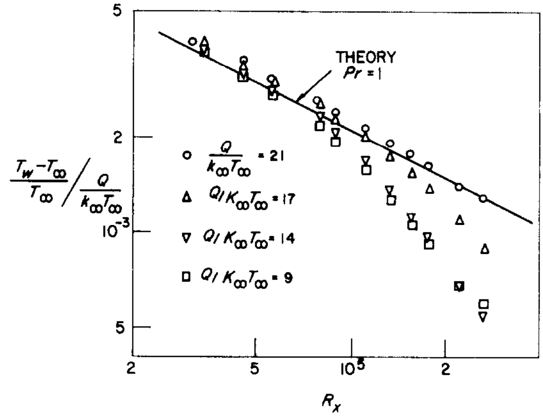

FIG. 11. Deviations from laminar wall temperatures for varying strengths of the heat source.

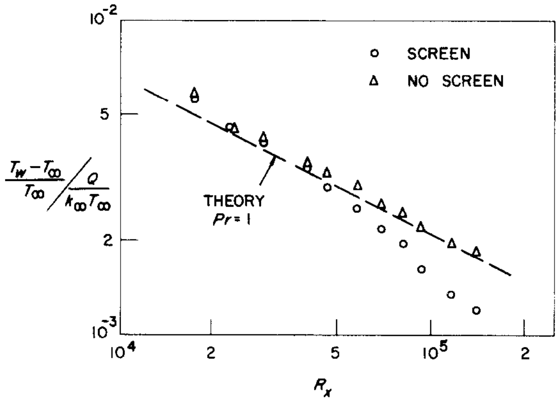

FIG. 12. Effect of a turbulent screen placed in front of the model.

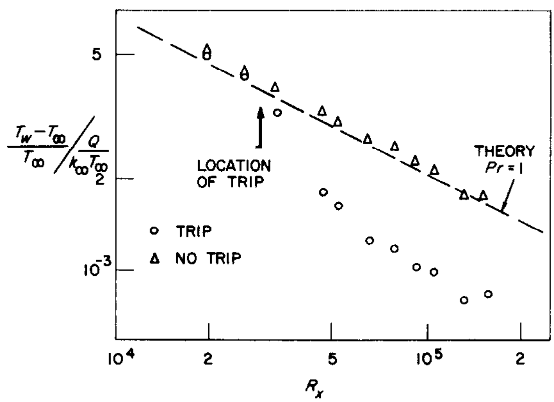

FIG. 13. Effect of a boundary-layer trip on the surface of the model.

provided the most striking illustration of the effect of transition. After the turbulent wall-temperature data had been recorded, the trip was removed without disturbing the free-stream flow. As the boundary layer returned to a laminar state, the wall temperature was observed to rise to the laminar values shown.

The observed behavior of the wall temperature means that one or more of the assumptions which permitted reasonable prediction of laminar values are somehow invalidated by turbulence. These assumptions are $Pr = 1$, $\mu \propto T$, $p =$ constant, and two-dimensional flow. The laminar analysis shows that only the magnitude of the wall temperature depends upon Prandtl number, and this dependence is relatively weak. The effects of non-linear viscosity variation are also relatively weak, at least for the limiting case of $T_w \gg T_\infty$, i.e. $S \gg 1$. By assuming $\mu \propto (T - T_\infty)^n$, which is a good approximation for $S \gg 1$, McCroskey [7] showed that T and τ have different Reynolds-number dependence; namely,

$$T - T_\infty \propto (R_x)^{-\frac{1}{n+1}}$$

$$\tau \propto (R_x)^{-\frac{n}{n+1}}.$$

Typically, $n = 0{\cdot}7$ to $0{\cdot}8$ for air, so that in the extreme case of $S \gg 1$, the Reynolds number dependence of T_w and τ is only slightly altered by non-linear viscosity variation.

The above remarks are based on laminar analyses. However, it does not seem likely that the transition mechanism would depend critically upon whether air, with $Pr = 0{\cdot}7$ and non-linear viscosity–temperature characteristics, or some other fluid, having $Pr = 1$ and $\mu \propto T$, were to be used in the experiments. Therefore it is felt that the radical departure from theory is not due to fluid properties such as these.

The constant-pressure assumption was not strictly fulfilled in the experiment, due to a small gradient $\partial p/\partial x$. However, varying this gradient by changing angle of attack and nose geometry did not alter the test results. Furthermore, pressure gradients are allowable, if the fluid is

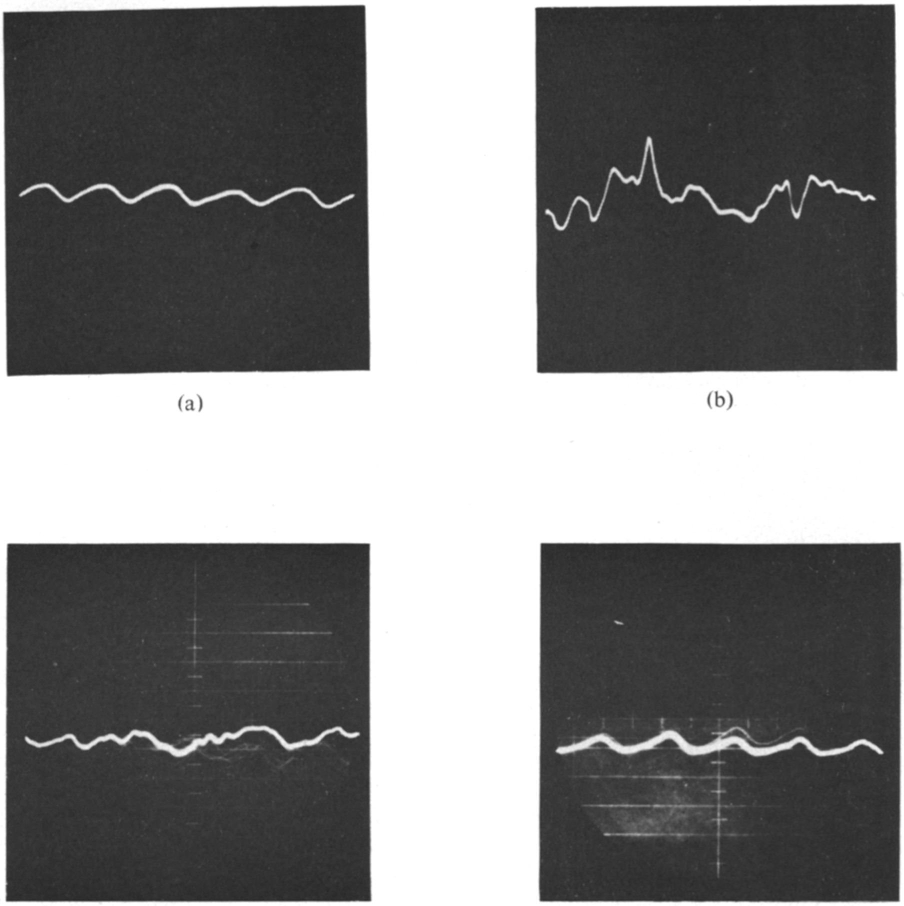

FIG. 14. Oscilloscope photographs of hot-wire fluctuations with increasing strengths of the heat source. Hot wire located at $x = 5$ in, $R_x = 85000$.

(a) Laminar boundary layer, 120 c/s electrical reference wave.
(b) Turbulence induced by trip at $x = 1$ in, $Q/k_\infty T_\infty = 0$.
(c) Partial suppression of fluctuations, $Q/k_\infty T_\infty = 25$.
(d) Complete suppression of trip-induced fluctuations, $Q/k_\infty T_\infty = 35$.

incompressible. The constant-property, time-dependent boundary-layer equations reveal by inspection that the pressure gradient should be proportional to the heat transfer at the wall for the temperature–shear-stress analogy to hold (cf. Meecham [3]). This result neglects the effect of the work of compression, dp/dt, which is zero only if $\rho =$ constant. It should be noted that no such difficulties regarding pressure gradients or fluctuations arise in the case of the *temperature–vorticity* analogy, but this requires constant fluid properties. However, it seems unlikely that compressibility effects would play a dominant role in the present experiments at low speeds and small S. Therefore, it is felt that variations in pressure were probably of secondary importance.

The remaining assumption, believed to be of primary importance in the failure of the theory, is the two-dimensional assumption. Although valid for laminar flows, this assumption is by no means realized in the chaotic motion of fluid particles in a turbulent layer. The role of these fluctuations is probably much different for shear stress or vorticity than for temperature. Vortex lines tend to be stretched laterally so as to maintain large values of vorticity and vorticity gradient near the wall, whereas temperature profiles are influenced more by the mechanism of turbulent diffusion. The conclusion, then, is that a two-dimensional description of turbulent flow is totally inadequate for the present problem.

Transition

The transitional stability of the boundary layer is, of course, beyond the scope of the present theoretical analysis. However, the experimental results are of interest as they show that the onset of turbulence is delayed by an increase in the strength of the heat source. This striking phenomenon was observed in both natural and forced transition.

It can be seen from the wall-temperature data in Fig. 11 that the boundary layer near the rear of the smooth plate was changed from turbulent to completely laminar as Q was varied by a factor of $2\frac{1}{2}$. Observations were also made downstream of a single-element trip tape as Q was varied. The hot-wire results are shown in Fig. 14. Again the boundary layer was changed from a transitional state to a laminar state by the addition of sufficient heat.

Tests were also made with uniformly distributed roughness on the plate. Transition Reynolds number was determined as S was varied, following the procedure described previously. The transition data in Fig. 8 were normalized by dividing R_{x_T} by the R'_{T_0} appropriate for the particular location on the plate (Fig. 7). The results, shown in Fig. 8, indicate an increase in R_{x_T} of about 15–20 per cent as S increased from zero to about 0·11.

The observed stabilizing influence of heat addition at the leading edge contrasts with the more common destabilizing influence of heat addition from the wall downstream of the leading edge. However, there is a fundamental difference in the two cases that should be pointed out. It is well-known that the velocity profile always possesses an inflection point, which tends to destabilize the flow, when heat is transferred to the fluid from a hot wall. On the other hand, the inflection point is at the wall, and not within the velocity profile, for the case of a boundary layer on an insulated flat plate which is heated either by viscous dissipation or by a point source of energy at the leading edge. In this connection, it may be noted that the experiments of Frick and McCullough [8] on a heated airfoil showed that heating the upper surface of the airfoil caused an inflection point in the laminar velocity profile and a subsequent reduction in transition Reynolds number, while heating only the nose produced no inflection point and no reduction in transition Reynolds number.

Although point-source heating is different from conventional heat transfer across a wall, point-source heating does produce effects which are strikingly similar to aerodynamic heating by viscous dissipation alone. It should be interesting to perform a classical stability analysis for the present source-heated boundary layer and to compare the result with the conventional flat-plate stability theory over a wide range of M and S. Locally, the temperature and velocity profiles are approximately the same, and since the classical stability theory depends only upon local profiles by virtue of the so-called "parallel-flow" assumption, it is quite possible that M and S would play analogous roles in determining $R_{x_{cr}}$.

Finally, it should be noted that an experimental program using an insulated flat plate with a point heat source at the leading edge offers the advantage of simplicity in comparison with conventional supersonic transition experiments since a subsonic wind tunnel can be used. Even more important is the fact that independent adjustment of Q and x allows the simulation of local velocity and temperature profiles, displacement thickness, and momentum thickness over a wide range of equivalent Mach number without changing the model or free stream conditions, such as turbulence level, tunnel pressure, and tunnel interference.

5. SUMMARY AND CONCLUSIONS

A specific test case for the temperature–shear-stress analogy has been investigated theoretically and experimentally. The temperature–shear-stress analogy, which is the compressible counterpart of the temperature–vorticity analogy for constant property flows, was verified in the laminar case by experiments which established the variation of wall temperature with the parameter $S = (Q/k_\infty T_\infty)R_x^{-\frac{1}{2}}$. On the other hand, the transition to turbulent flow was accompanied by significant decreases in the measured wall temperature, rather than the increase that would be expected on the basis of the behavior of either shear stress or vorticity. This suggests that transition to turbulence invalidates one or more of the following assumptions: $Pr = 1$, $\mu \propto T$, $p =$ constant, two-dimensional flow. The two-dimensional assumption appears to be the most significant one, particularly since it is crucial in both analogies.

Addition of heat by means of the point heat source was found to delay the onset of transition to turbulence, as opposed to the effect of a heated wall upon a conventional compressible boundary layer. Transition Reynolds number was found to increase with increasing S, and this striking phenomenon was observed in both natural and forced transition.

Finally, the laminar velocity and temperature profiles for the present physical problem simulate with good accuracy the profiles for a conventional flat-plate boundary layer heated by viscous dissipation, at a free stream Mach number of approximately $2\sqrt{S}$.

REFERENCES

1. H. SCHLICHTING, *Boundary Layer Theory*, 4th edn, p. 61. McGraw-Hill, New York (1960).
2. S. LEVY, Heat transfer to constant-property laminar boundary-layer flows with power-function free-stream velocity and wall-temperature variation, *J. Aeronaut. Sci.* **19**, 341–348 (1952).
3. W. C. MEECHAM, More general solutions of the incompressible boundary layer equation ($Pr = 1$), *AIAA Jl* **3**, 1192 (1965).
4. S. H. LAM, Shock-induced unsteady laminar compressible boundary layers on a semi-infinite flat plate. Ph.D. Thesis, Princeton University (1957).
5. N. RILEY, On the analogy between the transport of vorticity and heat in laminar boundary layers, *J. Fluid Mech.* **14**, 399–404 (1962).
6. E. R. VAN DRIEST, Investigation of laminar boundary layer in compressible fluids using the Crocco method, NACA TN 2597 (1952).
7. W. J. MCCROSKEY, Point heat sources and the temperature-vorticity analogy in compressible boundary layers. Princeton University Dept. of Aero. Eng. Report No. 613 (1962).
8. C. W. FRICK, JR. and G. B. MCCULLOUGH, Tests of a heated low-drag airfoil. NACA Wartime Report A-40 (1942).

Résumé—Sous certaines conditions acceptées généralement, l'analogie entre la température et la vorticité pour l'écoulement bidimensionnel d'un fluide visqueux avec propriétés constantes peut être étendue aux écoulements de couche limite compressible. Dans les limites des hypothèses, la température est reliée linéairement à la contrainte de cisaillement plutôt qu'à la vorticité. Les conséquences de cette relation sont discutées à la fois pour des écoulements laminaires et turbulents. L'analogie température–contrainte de cisaillement est analysée pour l'écoulement sur une plaque plane semi-infinie avec une source de chaleur ponctuelle d'intensité Q placée au bord d'attaque. Les profils de vitesse et de température sont donnés pour l'écoulement laminaire compressible. Ils ne sont pas en similitude, et dépendent du paramètre:

$$S = (Q/K_\infty T_\infty) R_x^{-\frac{1}{2}}.$$

De plus, ils ressemblent aux profils d'une couche limite compressible sur une plaque plane isolée avec $M_\infty^2 \propto S$ et sans source de chaleur.

Des résultats expérimentaux ont été obtenus qui sont en bon accord avec les calculs théoriques pour les écoulements laminaires. Cependant, les résultats ne suivent pas les prévisions de l'analogie température–vorticité dans les régimes de transition et turbulent, démontrant ainsi l'insuffisance des hypothèses utilisées dans la théorie. On a également trouvé que le nombre de Reynolds de début de transition pouvait être réglé en variant l'intensité de la source de chaleur, l'addition de chaleur par la source ponctuelle servant à retarder la transition à la turbulence.

On suggère qu'il peut y avoir une relation importante entre les effets du nombre de Mach et de l'addition de chaleur par une source ponctuelle sur les caractéristiques de stabilité et de transition des couches limites compressibles.

Zusammenfassung—Unter bestimmten, allgemein üblichen Annahmen kann die Analogie zwischen Temperatur und Verwirbelung einer zähen, zweidimensionalen Flüssigkeitsströmung mit konstanten Stoffwerten auf kompressible Grenzschichtströmungen erweitert werden. Im Bereich der Annahmen ist die Temperatur eher mit der Schubspannung als mit der Verwirbelung linear verknüpft. Die Folgen dieser Verknüpfung werden sowohl für laminare als auch turbulente Strömungen diskutiert. Die Temperatur–Schubspannungsanalogie wird für die Strömung über eine halbunendliche ebene Platte analysiert, wobei eine punktförmige Wärmequelle der Ergiebigkeit Q an der Anströmkante liegt. Geschwindigkeits- und Temperaturprofile werden für kompressible Laminarströmung angegeben. Sie sind nicht ähnlich und hängen vom Parameter

$$S = (Q/k_\infty T_\infty) R_x^{-\frac{1}{2}}.$$

ab. Ausserdem gleichen sie den Profilen einer kompressiblen Grenzschicht an einer isolierten ebenen Platte mit $M_\infty^2 \propto S$ und ohne Wärmequelle.

Die erhaltenen experimentellen Ergebnisse stimmen gut mit den theoretischen Berechnungen für Laminarströmung überein. Im Übergangs- und Turbulenzbereich der Strömung folgen die Daten nicht den Voraussagen der Temperatur–Verwirbelungs-Analogie, wodurch die Unzulänglichkeit der für die Theorie gemachten Annahmen demonstriert wird. Es zeigte sich, dass die Reynoldszahl bei welcher ein Umschlag erfolgte, durch Veränderung der Ergiebigkeit der Wärmequelle eingestellt werden konnte. Die Zunahme bewirkte dabei eine Verzögerung des Umschlags in Turbulenz. Es wird angedeutet, dass möglicherweise ein wichtiger Zusammenhang zwischen den Einflüssen der Machzahl und der punktförmigen Wärmezufuhr auf die Stabilität und die Übergangscharakteristik kompressibler Grenzschichten besteht.

Аннотация—При некоторых обычно принимаемых допущениях аналогию между температурой и интенсивностью вихрей для вязкого двумерного течения жидкости с постоянными свойствами можно развить на случай течения сжимаемой жидкости в пограничном слое. В пределах принятых допущений температура находится в линейной зависимости скорее от касательного напряжения, нежели от интенсивности вихрей. Обсуждается значение такого соотношения как для ламинарного, так и турбулентного потоков. Анализируется аналогия между температурой и касательным напряжением для случая обтекания полубесконечной плоской пластины с точеным источником тепла мощностью Q, расположенным на передней кромке. Для сжимаемого ламинарного потока представлены профили скорости и температуры. Они являются неподобными и зависят от параметра

$$S = (Q/k_\infty T_\infty) R_x^{-\frac{1}{2}}$$

Кроме того, они напоминают профили сжимаемого пограничного слоя на изолированной плоской пластине при $M_\infty^2 \propto S$ и отсутствии источника тепла.

Полученные экспериментальные результаты хорошо согласуются с теоретическими расчётами для ламинарных течений. Однако, они не совпадают со значениями, рассчитанными по температурно–вихревой аналогии при переходных и турбулентных режимах течения, тем самым указывая на неадэкватность предложений, сделанных в теории.

Найдено также, что критерий Рейнольдса, при котором начинается переход, можно подобрать, изменяя мощность источника тепла, т.е. источник тепла замедляет переход к турбулентности. Высказано предположение о существовании важной зависимости между влиянием критерия Маха и подводом тепла к точечному источнику на характеристики устойчивости и перехода сжимаемых пограничных слоев.